The multiplicative product $(\delta(x_0 - |x|)/|x|^{(n-2)/2})(\delta(x_0 + |x|)/|x|^{(n-2)/2})$

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In this paper we give a sense to the distributional multiplicative product $(\delta(x_0 - |x|)/|x|^{(n-2)/2})(\delta(x_0 + |x|)/|x|^{(n-2)/2})$. For the case $n = 4$ we obtain a formula which is used for a perturbative calculation of Green function in quantum field theories. As an application of our product, we show the calculation of the self-energy of the theory of $\lambda\phi^4$.

1. Introduction

Let $P(x_0, x_1, \dots, x_{n-1})$ be any sufficiently smooth function such that on $P = P(x_0, x_1, \dots, x_{n-1}) = 0$ we have $\operatorname{grad} P \neq 0$ (which means that there are no singular points on $P = 0$). Then the generalized function $\delta^{(k-1)}(P)$ is defined in [3, p. 211] by

$$\left(\delta^{(k-1)}(P), \varphi\right) = (-1)^{k-1} \int \Psi_{u_0}^{(k-1)}(0, u_1, \dots, u_{n-1})\, \mathrm{d}u_1 \dots \mathrm{d}u_{n-1}, \tag{1.1}$$

where

$$u_0 = P \tag{1.2}$$

and we choose the remaining u_i coordinates (with $i = 1, \dots, n-1$) arbitrarily except that the Jacobian of the x_i with respect to the u_i which we shall denote by $D\binom{x}{u}$ fails to vanish (which is always possible so long as $\operatorname{grad} P \neq 0$ on $P = 0$).

In (1.1),

$$\Psi(u) = \Psi(u_0, \dots, u_{n-1}) = \varphi_1 D\begin{pmatrix} x \\ u \end{pmatrix}, \tag{1.3}$$

$$\varphi_1(u_0, u_1, \dots, u_{n-1}) = \varphi(x_0, x_1, \dots, x_{n-1}) \tag{1.4}$$

and the integral of (1.1) is taken over the $P = 0$ surface, where $\varphi(x_0, x_1, \dots, x_{n-1})$ is an infinitely differentiable function with bounded support.

$$P(x_0, x_1, \dots, x_{n-1}) = x_0 - |x|, \tag{1.5}$$

where

$$|x| = \sqrt{x_1^2 + \cdots + x_{n-1}^2}. \tag{1.6}$$

From (1.1) we have

$$\big(\delta^{(k-1)}(P), \varphi\big) = (-1)^{k-1} \int \left[\frac{\partial^{k-1}}{\partial u_0^{k-1}} \left\{ \varphi(u_0, u_1, \ldots, u_{n-1}) \times D\begin{pmatrix} x \\ u \end{pmatrix} \right\} \right]_{u_0=0} \mathrm{d}u_1 \ldots \mathrm{d}u_{n-1}, \tag{1.7}$$

where

$$u_0 = P, \qquad u_1 = x_1, \qquad \ldots, \qquad u_{n-1} = x_{n-1}. \tag{1.8}$$

Let us transform to polar coordinates defined by

$$u_1 = s\omega_1, \qquad u_2 = s\omega_2, \qquad \ldots, \qquad u_{n-1} = s\omega_{n-1},$$

where

$$s = \sqrt{u_1^2 + \cdots + u_{n-1}^2}. \tag{1.9}$$

In these coordinates the element of volume is given by

$$\mathrm{d}x = \mathrm{d}x_0 \, \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} = \mathrm{d}x_0 \, s^{n-2} \, \mathrm{d}s \, \mathrm{d}\Omega_{n-1}, \tag{1.10}$$

where $\mathrm{d}\Omega_{n-1}$ is the element of surface area on the unit sphere in R^{n-1}. Then equation (1.5) becomes

$$P(x_0, x_1, \ldots, x_{n-1}) = x_0 - s. \tag{1.11}$$

Now let us choose the coordinates of P, x_0 and the ω_i. In these coordinates equation (1.10) becomes

$$\mathrm{d}x = \mathrm{d}x_0 \, (x_0 - P)^{n-2} \, \mathrm{d}s \, \mathrm{d}\Omega_{n-1}.$$

Then we may rewrite the defining equation (1.7) in the form

$$\big(\delta^{(k-1)}(P), \varphi\big) = (-1)^{k-1} \int \left[\frac{\partial^{k-1}}{\partial P^{k-1}} \left\{ \varphi (x_0 - P)^{n-2} \right\} \right]_{P=0} \mathrm{d}\Omega_{n-1} \, \mathrm{d}x. \tag{1.12}$$

Further, if we transform for P to $s = x_0 - P$ and note that $\partial/\partial P = (-1)\partial/\partial s$ we may write this in the form

$$\big(\delta^{(k-1)}\big(x_0 - |x|\big), \varphi\big) = \int \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \left\{ \varphi s^{n-2} \right\} \right]_{s=x_0} \mathrm{d}\Omega_{n-1} \, \mathrm{d}x_0. \tag{1.13}$$

Let us now write

$$\Psi(x_0, s) = \int_\Omega \varphi \, \mathrm{d}\Omega_{n-1}, \tag{1.14}$$

which transforms (1.13) to the form

$$\left(\delta^{(k-1)}\big(x_0-|x|\big),\varphi\right)=\int_0^\infty\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{\Psi(x_0,s)s^{n-2}\right\}\right]_{x_0=s}\mathrm{d}x_0. \tag{1.15}$$

Similarly, for

$$P(x_0,x_1,\ldots,x_{n-1})=x_0+|x|, \tag{1.16}$$

from (1.1) and (1.3) and considering (1.7) and (1.9) we obtain

$$\left(\delta^{(k-1)}\big(x_0+|x|\big),\varphi\right)=(-1)^{k-1}\int_0^{-\infty}\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{\Psi(x_0,s)s^{n-2}\right\}\right]_{s=-x_0}\mathrm{d}x_0. \tag{1.17}$$

2. The generalized functions $(x_0-|x|)_+^\lambda$ and $(x_0+|x|)_-^\lambda$

We define the generalized function $(x_0-|x|)_+^\lambda$, where λ is a complex number, by

$$\left(\big(x_0-|x|\big)_+^\lambda,\varphi\right)=\int_{x_0-|x|>0}\big(x_0-|x|\big)^\lambda\varphi|x|\,\mathrm{d}x, \tag{2.1}$$

where

$$x_0-|x|=x_0-\sqrt{x_1^2+\cdots+x_{n-1}^2}, \tag{2.2}$$

$x=(x_0,x_1,\ldots,x_{n-1})$ and $\mathrm{d}x=\mathrm{d}x_0\,\mathrm{d}x_1\ldots\mathrm{d}x_{n-1}$.

For $\operatorname{Re}\lambda\geqslant 0$, this integral converges and is an analytic function of λ. Analytic continuation to $\operatorname{Re}\lambda<0$ can be used to extend the definition of $((x_0-|x|)_+^\lambda,\varphi)$.

From (2.1) and considering the transform to polar coordinates (see equation (1.9)) we may rewrite equation (2.1) as

$$\left(\frac{(x_0-|x|)_+^\lambda}{|x|^{(n-2)/2}},\varphi\right)=\int_{x_0-s\geqslant 0}(x_0-s)^\lambda\varphi(x_0,s\omega)s^{(n-2)/2}\,\mathrm{d}x_0\,\mathrm{d}\omega\,\mathrm{d}s. \tag{2.3}$$

Putting

$$\Psi(x_0,s)=\int_\Omega\varphi(x_0,s\omega)\,\mathrm{d}\omega, \tag{2.4}$$

we have

$$\left(\frac{(x_0-|x|)_+^\lambda}{|x|^{(n-2)/2}},\varphi\right)=\int_0^\infty\int_0^{x_0}(x_0-s)^\lambda\Psi(x_0,s)s^{(n-2)/2}\,\mathrm{d}s\,\mathrm{d}x_0. \tag{2.5}$$

From (2.5) we have

$$\left(\frac{(x_0-|x|)_+^\lambda}{|x|^{(n-2)/2}},\varphi\right)=\int_0^\infty\int_0^{x_0}(x_0-s)^\lambda\Psi(x_0,s)s^{(n-2)/2}\,\mathrm{d}s\,\mathrm{d}x_0. \tag{2.6}$$

$\Psi(x_0, s)$ is an infinitely differentiable function of x_0 and s with bounded support. We now make the change of variables

$$s = x_0 \ell \tag{2.7}$$

in the integral (2.6), writing

$$\Psi(x_0, s) = \Psi_1(x_0, x_0\ell) \tag{2.8}$$

to obtain

$$\left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right) = \int_0^\infty \int_0^1 x_0^{\lambda+n/2} (1-\ell)^\lambda \ell^{(n-2)/2} \Psi_1(x_0, x_0\ell)\, \mathrm{d}\ell\, \mathrm{d}x_0. \tag{2.9}$$

This equation shows that $((x_0 - |x|)_+^\lambda / |x|^{(n-2)/2}, \varphi)$ has two sets of poles. The first of these consists of the poles of

$$G(\lambda, x_0) = \int_0^1 (1-\ell)^\lambda \ell^{(n-2)/2} \Psi_1(x_0, x_0\ell)\, \mathrm{d}\ell. \tag{2.10}$$

Taking into account that [3, p. 49]

$$\operatorname*{Res}_{\substack{\lambda=-k \\ k=1,2,\dots}} \left(x_+^\lambda, \varphi \right) = \frac{\varphi^{(k-1)}(0)}{(k-1)!}, \tag{2.11}$$

from (2.10) we have

$$\operatorname*{Res}_{\substack{\lambda=-k \\ k=1,2,\dots}} G(\lambda, x_0) = \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial \ell^{k-1}} \left\{ \ell^{(n-2)/2} \Psi_1(x_0, x_0\ell) \right\} \right]_{\ell=1}. \tag{2.12}$$

On the other hand,

$$\left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right) = \int_0^\infty x_0^{\lambda+n/2} G(\lambda, x_0)\, \mathrm{d}x_0 \tag{2.13}$$

may also have poles. This occurs at $\lambda = -n/2 - 1, -n/2 - 2, \dots$.

At these points

$$\operatorname*{Res}_{\substack{\lambda=-n/2-j \\ j=1,2,\dots}} \left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi \right) = \frac{1}{(j-1)!} \left[G^{(j-1)}\left(-\frac{n}{2} - j, x_0 \right) \right]_{x_0=0}. \tag{2.14}$$

Consequently, $((x_0 - |x|)_+^\lambda / |x|^{(n-2)/2}, \varphi)$ has two set of singularities, namely,

$$\lambda = -1, -2, \dots \tag{2.15}$$

and

$$\lambda = -\frac{n}{2} - 1, -\frac{n}{2} - 2, \dots, -\frac{n}{2} - j, \dots, \quad j = 1, 2, \dots. \tag{2.16}$$

Let us now study the case when

$$\lambda = -k, \quad k = 1, 2, \ldots, \tag{2.17}$$

and

$$\lambda \neq -\frac{n}{2} - j, \quad j = 1, 2, \ldots; \tag{2.18}$$

this is always the case when the dimension n is odd but is also true if n is even and $k < n/2 + 1$.

Let us write (2.10) in the neighborhood of $\lambda = -k$ in the form

$$G(\lambda, x_0) = \frac{G_0(x_0)}{\lambda + k} + G_1(\lambda, x_0), \tag{2.19}$$

where

$$G_0(x_0) = \operatorname*{Res}_{\lambda=-k} G(\lambda, x_0), \tag{2.20}$$

and $G_1(\lambda, x_0)$ is regular at $\lambda = -k$.

Inserting this into (2.13), we obtain

$$\left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \frac{1}{\lambda + k} \int_0^\infty x_0^{\lambda+n/2} G_0(x_0)\, \mathrm{d}x_0 + \int_0^\infty x_0^{\lambda+n/2} G_1(\lambda, x_0)\, \mathrm{d}x_0. \tag{2.21}$$

Under the assumptions we have made concerning λ, the integrals in (2.21) are regular functions of λ at $\lambda = -k$. Therefore $((x_0 - |x|)_+^\lambda / |x|^{(n-2)/2}, \varphi)$ has a simple pole at such a point and

$$\operatorname*{Res}_{\substack{\lambda=-k\\ k=1,2,\ldots}} \left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \int_0^\infty x_0^{n/2-k} G_0(x_0)\, \mathrm{d}x_0, \tag{2.22}$$

where for $k \geqslant n/2 + 1$ the integral is understood in the sense of its regularization (see [3, chapter I, section 3]).

Inserting equation (1.12) for $G_0(x_0)$, we arrive at

$$\operatorname*{Res}_{\lambda=-k} \left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^\infty x_0^{n/2-k} \times \left[\frac{\partial^{k-1}}{\partial \ell^{k-1}} \left\{\ell^{(n-2)/2} \Psi_1(x_0, x_0\ell)\right\}\right]_{\ell=1} \mathrm{d}x_0, \tag{2.23}$$

where $\Psi_1(x_0, x_0\ell)$ is defined by (2.8).

Note that if we write $x_0\ell = s$, we obtain

$$\left[\frac{\partial^{k-1}}{\partial \ell^{k-1}} \left\{\ell^{(n-2)/2} \Psi_1(x_0, x_0\ell)\right\}\right]_{\ell=1} = x_0^{k-1-n/2+1} \frac{(-1)^{k-1}}{(k-1)!} \times \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \left\{s^{(n-2)/2} \Psi_1(x_0, s)\right\}\right]_{s=x_0}, \tag{2.24}$$

so that we may rewrite (2.23) in the form

$$\operatorname*{Res}_{\substack{\lambda=-k\\ k=1,2,\dots}}\left(\frac{(x_0-|x|)_+^\lambda}{|x|^{(n-2)/2}},\varphi\right)=\frac{(-1)^{k-1}}{(k-1)!}\times\int_0^\infty\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{s^{(n-2)/2}\Psi_1(x_0,s)\right\}\right]_{s=x_0}\mathrm{d}x_0. \quad (2.25)$$

Let us now study $(x_0+|x|)_-^\lambda$. We define the generalized function $(x_0+|x|)_-^\lambda$, where λ is a complex number, by

$$\left((x_0+|x|)_-^\lambda,\varphi\right)=\int_{-(x_0+|x|)>0}\left(-(x_0+|x|)\right)^\lambda\varphi(x)\,\mathrm{d}x. \quad (2.26)$$

As in case for $(x_0-|x|)_+^\lambda$ we arrive at the following result analogous to (2.9) and (2.10):

$$\left(\frac{(x_0+|x|)_-^\lambda}{|x|^{(n-2)/2}},\varphi\right)=\int_0^{-\infty}(-x_0)^{\lambda+n/2}G(\lambda,-x_0)\,\mathrm{d}x_0, \quad (2.27)$$

where

$$G(\lambda,-x_0)=\int_0^1(1-\ell)^\lambda\ell^{(n-2)/2}\Psi_1(x_0,-x_0\ell)\,\mathrm{d}\ell. \quad (2.28)$$

From (2.27), (2.28), considering (2.11)–(2.14), we have

$$\operatorname*{Res}_{\substack{\lambda=-j\\ j=1,2,\dots}}G(\lambda,-x_0)=\frac{(-1)^{j-1}}{(j-1)!}\left[\frac{\partial^{j-1}}{\partial\ell^{j-1}}\left\{\ell^{(n-2)/2}\Psi_1(x_0,-x_0\ell)\right\}\right]_{\ell=1} \quad (2.29)$$

and

$$\operatorname*{Res}_{\lambda=-n/2-j}\left(\frac{(x_0+|x|)_-^\lambda}{|x|^{(n-2)/2}},\varphi\right)=(-1)\frac{1}{(j-1)!}\left[G^{(j-1)}\left(-\frac{n}{2}-j,-x_0\right)\right]_{x_0=0}. \quad (2.30)$$

Also $((x_0+|x|)_-^\lambda/|x|^{(n-2)/2},\varphi)$ has two sets of singularities,

$$\lambda=-1,-2,\dots\quad\text{and}\quad\lambda=-\frac{n}{2}-1,-\frac{n}{2}-2,\dots, \quad (2.31)$$

and considering equations (2.19)–(2.24), we arrive at the following result analogous to (2.25):

$$\operatorname*{Res}_{\substack{\lambda=-k\\ k=1,2,\dots}}\left(\frac{(x_0+|x|)_-^\lambda}{|x|^{(n-2)/2}},\varphi\right)=\frac{(-1)^{j-1}}{(j-1)!}\times\int_0^{-\infty}\left[\frac{\partial^{j-1}}{\partial s^{j-1}}\left\{s^{(n-2)/2}\Psi_1(x_0,s)\right\}\right]_{s=-x_0}\mathrm{d}x_0. \quad (2.32)$$

Putting $k = 1$ in (2.25) and $j = 1$ in (2.32), we obtain

$$\operatorname*{Res}_{\lambda=-1}\left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \int_0^\infty \left[s^{(n-2)/2}\Psi_1(x_0, s)\right]_{s=x_0} \mathrm{d}x_0 \tag{2.33}$$

and

$$\operatorname*{Res}_{\lambda=-1}\left(\frac{(x_0 + |x|)_-^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \int_0^{-\infty} \left[s^{(n-2)/2}\Psi_1(x_0, s)\right]_{s=-x_0} \mathrm{d}x_0. \tag{2.34}$$

On the other hand, from (1.15) and (1.17) we have

$$\left(\frac{\delta(x_0 - |x|)}{|x|^{(n-2)/2}}, \varphi\right) = \int_0^\infty \left[\Psi(x_0, s)s^{(n-2)/2}\right]_{s=x_0} \mathrm{d}s \tag{2.35}$$

and

$$\left(\frac{\delta(x_0 + |x|)}{|x|^{(n-2)/2}}, \varphi\right) = \int_0^{-\infty} \left[\Psi(x_0, s)s^{(n-2)/2}\right]_{s=-x_0} \mathrm{d}s. \tag{2.36}$$

From (2.35) and (2.36) and considering (2.33) and (2.34) we obtain

$$\operatorname*{Res}_{\lambda=-1}\left(\frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \left(\frac{\delta(x_0 - |x|)}{|x|^{(n-2)/2}}, \varphi\right) \tag{2.37}$$

and

$$\operatorname*{Res}_{\lambda=-1}\left(\frac{(x_0 + |x|)_-^\lambda}{|x|^{(n-2)/2}}, \varphi\right) = \left(\frac{\delta(x_0 + |x|)}{|x|^{(n-2)/2}}, \varphi\right). \tag{2.38}$$

Summarizing, we have the following. For odd n and for even n if $k < n/2+1$, the generalized functions $(x_0 - |x|)_+^\lambda/|x|^{(n-2)/2}$ and $(x_0 + |x|)_-^\lambda/|x|^{(n-2)/2}$ have simple poles at $\lambda = -k$, $k = 1, 2, \ldots$, and the following formulae are valid:

$$\operatorname*{Res}_{\lambda=-1} \frac{(x_0 - |x|)_+^\lambda}{|x|^{(n-2)/2}} = \frac{\delta(x_0 - |x|)}{|x|^{(n-2)/2}} \tag{2.39}$$

and

$$\operatorname*{Res}_{\lambda=-1} \frac{(x_0 + |x|)_-^\lambda}{|x|^{(n-2)/2}} = \frac{\delta(x_0 + |x|)}{|x|^{(n-2)/2}}. \tag{2.40}$$

3. The generalized function $(x_0^2 - |x|^2)_-^\lambda$

We define the generalized function $(x_0^2 - |x|^2)_-^\lambda$, where λ is a complex number, by

$$\left((x_0^2 - |x|^2)_-^\lambda, \varphi\right) = \int_{-(x_0^2-|x|^2)\geqslant 0} \left(-(x_0^2 - |x|^2)\right)^\lambda \varphi(x)\, \mathrm{d}x, \tag{3.1}$$

where

$$x_0^2 - |x|^2 = x_0^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2, \tag{3.2}$$

$x = (x_0, x_1, \ldots, x_{n-1})$ and $\mathrm{d}x = \mathrm{d}x_0\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}$.

For $\operatorname{Re}\lambda \geqslant 0$, the functional $((x_0^2 - |x|^2)_-^\lambda, \varphi)$ corresponds to the function

$$\left(x_0^2 - |x|^2\right)_-^\lambda = \begin{cases} (-(x_0^2 - |x|^2))^\lambda & \text{if } x_0^2 - |x|^2 < 0, \\ 0 & \text{if } x_0^2 - |x| > 0. \end{cases} \tag{3.3}$$

From [3, p. 253], we know that integral (3.1) converges for $\operatorname{Re}\lambda \geqslant 0$ and is an analytic function of λ.

Analytic continuation to $\operatorname{Re}\lambda < 0$ can be used to extend the definition of $((x_0^2 - |x|^2)_-^\lambda, \varphi)$.

Now proceeding as in section 2, we obtain

$$\begin{aligned} \left(\frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}, \varphi\right) &= \int_{-x_0^2+|x|^2\geqslant 0} (-1)^\lambda \frac{(x_0^2 - |x|^2)^\lambda}{|x|^{n-2}} \varphi(x_0, \ldots, x_{n-1})\,\mathrm{d}x \\ &= (-1)^\lambda \int_0^\infty \int_0^s \frac{(x_0^2 - s^2)^\lambda}{s^{n-2}} \Psi(x_0, s) s^{n-2}\,\mathrm{d}x_0\,\mathrm{d}s \\ &= (-1)^\lambda \int_0^\infty \int_0^s \left(x_0^2 - s^2\right)^\lambda \Psi(x_0, s)\,\mathrm{d}x_0\,\mathrm{d}s, \end{aligned} \tag{3.4}$$

where $\Psi(x_0, s)$ is defined by (1.14).

We now make the change of variables

$$u = x_0^2, \qquad v = s^2 \tag{3.5}$$

in the integral of (3.4), writing

$$\Psi(x_0, s) = \Psi_1(u, v) \tag{3.6}$$

to obtain

$$\left(\frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}, \varphi\right) = \frac{1}{4} \int_0^\infty \int_0^v (-u + v)^\lambda \Psi_1(u, v) v^{-1/2} u^{-1/2}\,\mathrm{d}u\,\mathrm{d}v. \tag{3.7}$$

Finally, we write $u = vt$, which transforms (3.7) to the form

$$\left(\frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}, \varphi\right) = \int_0^\infty v^\lambda \Phi(\lambda, v)\,\mathrm{d}v, \tag{3.8}$$

where

$$\Phi(\lambda, v) = \frac{1}{4} \int_0^1 (1 - t)^\lambda t^{-1/2} \Psi_1(vt, v)\,\mathrm{d}t. \tag{3.9}$$

This equation shows that $((x_0^2 - |x|^2)_-^\lambda / |x|^{n-2}, \varphi)$ has singularities at

$$\lambda = -j, \quad j = 1, 2, \ldots. \tag{3.10}$$

Let us write (3.9) in the neighborhood of $\lambda = -j$, $j = 1, 2, \ldots$, in the form

$$\Phi(\lambda, v) = \frac{\Phi_0(v)}{\lambda + j} + \Phi_1(\lambda, v), \tag{3.11}$$

where

$$\Phi_0(v) = \operatorname*{Res}_{\lambda=-j} \Phi(\lambda, v) \tag{3.12}$$

and $\Phi_1(\lambda, v)$ is regular at $\lambda = -j$.

Inserting this into (3.8), we obtain

$$\left(\frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}, \varphi\right) = \frac{1}{\lambda + j} \int_0^\infty v^\lambda \Phi_0(v)\, \mathrm{d}u + \int_0^\infty v^\lambda \Phi_1(\lambda, v)\, \mathrm{d}u. \tag{3.13}$$

Each of the integral in (3.13) has a simple pole at this value of λ.

Therefore $((x_0^2 - |x|^2)_-^\lambda / |x|^{n-2}, \varphi)$ has a pole of order two at $\lambda = -j$, $j = 1, 2, \ldots$.

In the neighborhood of such a point we may expand $(x_0^2 - |x|^2)_-^\lambda / |x|^{n-2}$ in the Laurent series

$$\frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}} = \frac{A^j}{(\lambda + j)^2} + \frac{B^j}{\lambda + j} + \cdots. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned} \lim_{\lambda \to -j} \left((\lambda + j)^2 \frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}, \varphi\right) &= (A^j, \varphi) = \lim_{\lambda \to -j} (\lambda + j) \int_0^\infty v^\lambda \Phi_0(u)\, \mathrm{d}u \\ &= \operatorname*{Res}_{\substack{\lambda=-j \\ j=1,2,\ldots}} \int_1^\infty v^\lambda \Phi_0(u)\, \mathrm{d}u = \frac{\Phi_0^{(j-1)}(0)}{(j-1)!}. \end{aligned} \tag{3.15}$$

Putting $j = 1$ in (3.15) and considering (3.9) and (3.12), we have

$$\begin{aligned} \lim_{\lambda \to -1} \left((\lambda + 1)^2 \frac{(x_0^2 - |x|^2)_-^\lambda}{|x|^{n-2}}, \varphi\right) &= (A^1, \varphi) = \Phi_0(0) \\ &= \operatorname*{Res}_{\lambda=-1} \frac{1}{4} \int_0^1 (1-t)^\lambda t^{-1/2} \Psi_1(0,0)\, \mathrm{d}t. \end{aligned} \tag{3.16}$$

But

$$\int_0^1 (1-t)^\lambda t^{-1/2}\, \mathrm{d}t = \int_0^1 (1-t)^{\lambda+1-1} t^{1/2-1}\, \mathrm{d}t = \frac{\Gamma(\lambda+1)\Gamma(1/2)}{\Gamma(\lambda+1+1/2)}, \tag{3.17}$$

and from (3.6) and (1.14),

$$\Psi_1(0,0) = \Psi(0,0) = \int \varphi(0 \ldots 0)\, \mathrm{d}\omega = \Omega_{n-1} \varphi(0), \tag{3.18}$$

where

$$\Omega_{n-1} = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}. \tag{3.19}$$

From (3.16) and considering (3.17)–(3.19), we have

$$\lim_{\lambda\to-1}\left((\lambda+1)^2\frac{(x_0^2-|x|^2)_-^\lambda}{|x|^{n-2}},\varphi\right) = \frac{1}{2}\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\varphi(0)\operatorname*{Res}_{\lambda=-1}\Gamma(\lambda+1). \tag{3.20}$$

We take account that ([1, p. 344])

$$\operatorname*{Res}_{z=-m}\Gamma(z) = \frac{(-1)^m}{m!} \quad \text{for } m = 0,1,2,\ldots. \tag{3.21}$$

From (3.20), we have

$$\lim_{\lambda\to-1}\left((\lambda+1)^2\frac{(x_0^2-|x|^2)_-^\lambda}{|x|^{n-2}},\varphi\right) = \frac{1}{2}\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\varphi(0) \tag{3.22}$$

or, in other words, we obtain

$$\lim_{\lambda\to-1}\left((\lambda+1)^2\frac{(x_0^2-|x|^2)_-^\lambda}{|x|^{n-2}}\right) = \frac{1}{2}\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\delta(x_0,x_1,\ldots,x_{n-1}). \tag{3.23}$$

4. The multiplicative product of $\delta(x_0-|x|)/|x|^{(n-2)/2}$ and $\delta(x_0+|x|)/|x|^{(n-2)/2}$

For $\operatorname{Re}\lambda \geqslant 0$, the functionals $((x_0-|x|)_+^\lambda,\varphi)$ and $((x_0+|x|)_-^\lambda,\varphi)$ correspond to the functions

$$\left(x_0-|x|\right)_+^\lambda = \begin{cases}(x_0-|x|)^\lambda & \text{if } x_0-|x| \geqslant 0,\\ 0 & \text{if } x_0-|x| < 0,\end{cases} \tag{4.1}$$

and

$$\left(x_0+|x|\right)_-^\lambda = \begin{cases}(-(x_0-|x|))^\lambda & \text{if } x_0+|x| \leqslant 0,\\ 0 & \text{if } x_0+|x| > 0.\end{cases} \tag{4.2}$$

From (4.1) and (4.2), we have

$$\left(x_0-|x|\right)_+^\lambda\left(x_0+|x|\right)_-^\lambda = \left(x_0^2-|x|^2\right)_-^\lambda, \tag{4.3}$$

where $(x_0^2-|x|)_-^\lambda$ is defined by (3.3).

From (2.39) and (2.40), we obtain the following theorem:

Theorem. Let n be dimension of the space, then the following formula is valid:

$$\frac{\delta(x_0-|x|)}{|x|^{(n-2)/2}}\frac{\delta(x_0+|x|)}{|x|^{(n-2)/2}} = \frac{1}{2}\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\delta(x_0,x_1,\ldots,x_{n-1}). \tag{4.4}$$

Proof. From (2.39) and (2.40), we have

$$\begin{aligned}\frac{\delta(x_0-|x|)}{|x|^{(n-2)/2}}\frac{\delta(x_0+|x|)}{|x|^{(n-2)/2}} &= \operatorname*{Res}_{\lambda=-1}\frac{(x_0-|x|)_+^\lambda}{|x|^{(n-2)/2}}\operatorname*{Res}_{\lambda=-1}\frac{(x_0+|x|)_-^\lambda}{|x|^{(n-2)/2}}\\ &= \lim_{\lambda\to-1}(\lambda+1)\frac{(x_0-|x|)_+^\lambda}{|x|^{(n-2)/2}}\lim_{\lambda\to-1}(\lambda+1)\frac{(x_0+|x|)_-^\lambda}{|x|^{(n-2)/2}}\\ &= \lim_{\lambda\to-1}\left[(\lambda+1)^2\frac{(x_0-|x|)_+^\lambda(x_0+|x|)_-^\lambda}{|x|^{n-2}}\right]. \qquad (4.5)\end{aligned}$$

From (4.5) and (4.3) we have

$$\left(\frac{\delta(x_0-|x|)}{|x|^{(n-2)/2}}\frac{\delta(x_0+|x|)}{|x|^{(n-2)/2}},\varphi\right) = \lim_{\lambda\to-1}\left((\lambda+1)^2\frac{(x_0^2-|x|^2)_-^\lambda}{|x|^{n-2}},\varphi\right). \qquad (4.6)$$

From (4.6) and considering (3.20), we obtain

$$\left(\frac{\delta(x_0-|x|)}{|x|^{(n-2)/2}}\frac{\delta(x_0+|x|)}{|x|^{(n-2)/2}},\varphi\right) = \frac{1}{2}\frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\varphi(0). \qquad (4.7)$$

From (4.7) we conclude

$$\frac{\delta(x_0-|x|)}{|x|^{(n-2)/2}}\frac{\delta(x_0+|x|)}{|x|^{(n-2)/2}} = \frac{1}{2}\pi^{(n-1)/2}\frac{1}{\Gamma((n-1)/2)}\delta(x_0,x_1,\ldots,x_{n-1}). \qquad (4.8)$$

In particular if $n=4$, from (4.4) we obtain

$$\frac{\delta\left(x_0-\sqrt{x_1^2+x_2^2+x_3^2}\right)}{r}\frac{\delta\left(x_0+\sqrt{x_1^2+x_2^2+x_3^2}\right)}{r} = \pi\delta(x_0,x_1,x_2,x_3)$$

or, equivalently, multiplying the above equation by the factor $1/2$ we obtain

$$\begin{aligned}\frac{\delta\left(x_0-\sqrt{x_1^2+x_2^2+x_3^2}\right)\delta\left(x_0+\sqrt{x_1^2+x_2^2+x_3^2}\right)}{2r^2} &= \frac{\delta(x_0-r)\delta(x_0+r)}{2r^2}\\ &= \frac{\pi}{2}\delta(x_0,x_1,x_2,x_3), \qquad (4.9)\end{aligned}$$

where $r=\sqrt{x_1^2+x_2^2+x_3^2}$. □

The generalization of this result was inspired by formula (4.9) which could be applied for a perturbative calculation of Green function in quantum field theories.

5. Application

Using this product together with the result showed in [2], one can compute, for instance, the self-energy corresponding to a theory $\lambda\phi^4$.

The Langrangian of this theory is

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi + \frac{\lambda}{4}\phi^4 \ .$$

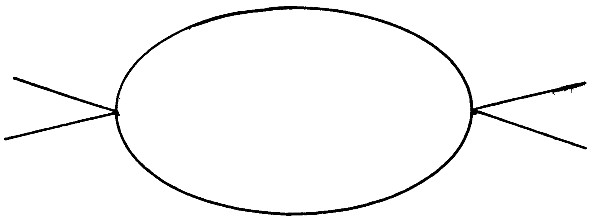
In the metric $(1,-1,-1,-1)$, the propagator is

$$\Delta(x) = \frac{1}{4\pi^2}\left(x^2 - \mathrm{i}0\right)^{-1}, \qquad x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Taking into account our result and [2], we get

$$\Delta^2(x) = \frac{1}{16\pi^4}\left(x^2 - \mathrm{i}0\right)^{-1}\left(x^2 - \mathrm{i}0\right)^{-1} = \frac{1}{16\pi^4}\left(x^2 - \mathrm{i}0\right)^{-2}.$$

This result corresponds to the knot which appears in the theory of self-energy:



In this case, the self-energy becomes

$$\sum(x) = \left(\frac{\lambda}{4}\right)^2\Delta^2(x) = \left(\frac{\lambda}{4}\right)^2\frac{1}{16\pi^4}\left(x^2 - \mathrm{i}0\right)^{-2}.$$

This way we obtain a known result directly, applying product of propagators.

References

[1] Bateman, *Higher Trascendental Functions*, Vol. 1 (Manuscript project, McGraw-Hill, 1953).
[2] C. Bollini and M. Rocca, Convolution of ultradistributions, Preprint, La Plata (1997, in preparation).
[3] I.M. Gelfand and G.E. Shilov, *Generalized Functions*, Vol. 1 (Academic Press, New York, 1964).